On the invalidity of Helmholtz's reciprocity theorem for Green's functions describing the propagation of a scalar wave field in a non empty- and empty space

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On the invalidity of Helmholtz's reciprocity theorem for Green's functions describing the propagation of a scalar wave field in a non empty- and empty space

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Abstract

It is shown by counterexamples that Green's functions describing the propagation of a field from one surface to another do not, in general, satisfy Helmholtz's reciprocity theorem.

Inhalt

Zur Ungültigkeit des Reziprozitätstheorems von Helmholtz für Greensche Funktionen zur Beschreibung der Fortpflanzung eines sakalaren Wellenfeldes in nichtleerem und leerem Raum. Wie durch Gegenbeispiele gezeigt wird, erfüllen Greensche Funktionen bei der Beschreibung der Fortpflanzung eines Feldes von einer Fläche zu einer anderen nicht das Reziprozitätstheorem von Helmholtz.

1. Introduction

The Helmholtz reciprocity theorem states in Helmholtz's own words that (Helmholtz [1]): Wenn in einem mit Luft gefüllten Raume, der teils von endlich ausgedehnten festen Körpern begrenzt, teils unbegrenzt ist, im Punkte a Schallwellen erregt werden, so ist das Geschwindigkeitspotential derselben in einem zweiten Punkte b ebenso groß, als es in a sein würde, wenn nicht in a, sondern in b Wellen von derselben Intensität erregt würden. Auch ist der Unterschied der Phasen des erregenden und erregten Punktes in beiden Fällen gleich.

If a space filled with air which is partly bounded by finitely extended fixed bodies and is partly unbounded, sound waves be excited at any point A, the resulting velocity-potential at a second point B is the same both in magnitude and phase, as it would have been at A, had B been the source of the sound.

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Helmholtz's reciprocity theorem is, from a modern point of view a direct consequence of the symmetry of a Green's function $G(\boldsymbol{r}, \boldsymbol{r}_0; k)$ generated by a self adjoint linear partial differential equation, like the Helmholtz equation, with a homogeneous boundary condition (Courant and Hilbert [2], Morse and Feshbach [3]):

$$(\nabla^2 + k^2 n^2(\boldsymbol{r}))\, G(\boldsymbol{r}_0, \boldsymbol{r}; k) = \delta(\boldsymbol{r} - \boldsymbol{r}_0), \tag{1.1}$$

if $\boldsymbol{r}$ and $\boldsymbol{r}_0 \in$ domain D and n denotes the scalar index of refraction. The Green's function G satisfies a homogeneous boundary condition if $\boldsymbol{r}$ is situated at the boundary of the domain D for all values of $\boldsymbol{r}_0 \in D$ and

$$G(\boldsymbol{r}, \boldsymbol{r}_0; k) = G(\boldsymbol{r}_0, \boldsymbol{r}; k), \text{ (reciprocity)}. \tag{1.2}$$

The symmetry property (1.2) is the mathematical expression of Helmholtz's reciprocity theorem, formulated above.

The Green's function G is the appropriated tool for the calculation of the field ψ satisfying

$$(\nabla^2 + k^2 n^2(\boldsymbol{r}))\, \psi(r) = \varrho(\boldsymbol{r}), \tag{1.3}$$

if $\boldsymbol{r} \in D$ and ψ satisfies certain homogeneous boundary conditions, viz

$$\psi(\boldsymbol{r}) = \int_D G(\boldsymbol{r}, \boldsymbol{r}'; k)\, \varrho(\boldsymbol{r}')\, d\boldsymbol{r}'. \tag{1.4}$$

However, the theory of Green's functions changes substantially if we have to construct Green's functions describing the propagation of a scalar field ψ satisfying

$$(\nabla^2 + k^2 n^2(\boldsymbol{r}))\, \psi(\boldsymbol{r}) = 0, \tag{1.5}$$

from a surface σ on which ψ attains prescribed values $\nu(\tau)$. Assuming that the field ψ satisfies Sommerfeld's radiation condition at infinity,

$$\lim_{r \to \infty} r \left(\frac{\partial \psi}{\partial r} - ik\psi \right) = 0, \tag{1.6}$$

it can be shown that Sommerfeld [4], § 10

$$\psi(r) = \int_\sigma \frac{\partial}{\partial n_\sigma} G(\boldsymbol{r}, \sigma; k)\, V(\sigma)\, d\sigma. \tag{1.7}$$

The kernel Green's function G is the (unique) solution of the equation

$$(\nabla^2 + k^2n^2(\boldsymbol{r}))\, G(\boldsymbol{r}, \boldsymbol{r}_0; k) = \delta(\boldsymbol{r} - \boldsymbol{r}_0), \tag{1.8}$$

Satisfying

$$G(\boldsymbol{r}, \boldsymbol{r}_0; k) = 0, \quad \text{if} \quad \boldsymbol{r}_0 \in \sigma \quad \forall \boldsymbol{r} \text{ outside } \sigma, \tag{1.9}$$

and Sommerfeld's radiation condition at infinity

$$\lim_{r\to\infty} r \left(\frac{\partial\varphi}{\partial r} G(\boldsymbol{r}, \boldsymbol{r}_0; k) - ikG(\boldsymbol{r}, \boldsymbol{r}_0; k) \right) = 0. \tag{1.10}$$

The differentiation $\dfrac{\partial}{\partial n_\sigma}$ has to be performed with respect to the variable σ.

We will, in view of eq. (1.7) call a function $\dfrac{\partial}{\partial n_\sigma} G(\boldsymbol{r}, \sigma; k)$ a surface Green's function and a function $G(\boldsymbol{r}, \boldsymbol{r}_0; k)$ a volume Green's function (Morse and Feshbach [3]). Consequently we are, in view of eq. (1.2) led to the following question: does a surface Green's function satisfy a reciprocity relation like the volume Green's function?

This problem has been considered by Butterweck [5] who, using an elegant analysis, showed that surface Green's functions satisfy a reciprocity relation in the large: *A surface Green's function is reciprocal if details of the order of magnitude of a few wavelengths are neglected.* Because the basic relation of his theory (eq. 27) does not provide an answer to the reciprocity question formulated above we will construct examples of non-reciprocal surface Green's functions in the following sections.

In section 2 we consider the propagation from plane to plane of a scalar wave field propagating in a space characterized by a scalar index of refraction $n^2 = \varepsilon\delta(\boldsymbol{r} - \boldsymbol{b}) + 1$. In section 3 we consider the propagation of a scalar wave field in *free* space from a spherical surface to another spherical surface.

2. Construction of a non-reciprocal surface Green's function

Let D denote the infinite half space to the right of the plane $z = 0$ and n the index of refraction of an object with finite support. Suppose that D' denote the half space to the left of the plane $z = a$. The Green's function $K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k)$ satisfying

$$(\nabla r^2 + k^2n^2(\boldsymbol{r}))K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) = \delta(\boldsymbol{r} - \boldsymbol{r}_0), \tag{2.1}$$

$$K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) = 0, \quad \text{if} \quad \boldsymbol{r}_0 \in z = 0 \quad \forall \boldsymbol{r} \in D, \tag{2.2}$$

and Sommerfeld's radiation condition at infinity

$$\lim_{r\to\infty} r\left(\frac{\partial}{\partial r} k_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) - ik\, K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k)\right) = 0, \tag{2.3}$$

if $\boldsymbol{r} \in D$ and $|\boldsymbol{r}| \to \infty$,

is the solution of the linear integral equation

$$K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) = G_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) + \int_\tau k^2(1 - n^2(\boldsymbol{r}_0')) \, G_{12}(\boldsymbol{r}, \boldsymbol{r}_0') \, K_{12}(\boldsymbol{r}_0', \boldsymbol{r}_0) \, d\boldsymbol{r}_0', \tag{2.4}$$

if τ denotes the support of the medium and

$$G_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) = |\boldsymbol{r} - \boldsymbol{r}_0|^{-1} \exp ik\,|\boldsymbol{r} - \boldsymbol{r}_0| - |\boldsymbol{r} - \tilde{\boldsymbol{r}}_0|^{-1} \exp ik\,|\boldsymbol{r} - \tilde{\boldsymbol{r}}_0|, \tag{2.5}$$

where $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ and $\tilde{\boldsymbol{r}}_0 = (x_0, y_0, -z_0)$. (2.6)

The index 12 indicates that we consider the propagation from the plane $z = 0$ to the plane $z = a$. The index 21 in the following formulae indicates that we consider propagation from the plane $z = a$ to the plane $z = 0$.

The Green's function $K_{21}(\boldsymbol{r}, \boldsymbol{r}_0; k)$ satisfying

$$(\nabla_0^2 + k^2 n^2(\boldsymbol{r}_0) \, K_{21}(\boldsymbol{r}, \boldsymbol{r}_0; k) = \delta(\boldsymbol{r} - \boldsymbol{r}_0), \tag{2.7}$$

$$K_{21}(\boldsymbol{r}, \boldsymbol{r}_0; k) = 0, \quad \text{if} \quad \boldsymbol{r} \in z = a, \quad \forall \boldsymbol{r}_0 \in D' \tag{2.8}$$

and Sommerfeld's radiation condition at infinity

$$\lim_{r_0\to\infty} r_0\left(\frac{\partial}{\partial r_0} K_{21}(\boldsymbol{r}, \boldsymbol{r}_0; k) - ikK_{21}(\boldsymbol{r}, \boldsymbol{r}_0; k)\right) = 0, \tag{2.9}$$

if $\boldsymbol{r}_0 \in D'$ and $|\boldsymbol{r}_0| \to \infty$,

is the solution of the linear integral equation

$$K_{21}(r, r_0; k) = G_{21}(\boldsymbol{r}, \boldsymbol{r}_0; k) + \int_\tau k^2(1 - n^2(\boldsymbol{r}_0') \, G_{21}(\boldsymbol{r}, \boldsymbol{r}_0'; k) \, K_{21}(\boldsymbol{r}_0', \boldsymbol{r}_0) \, d\boldsymbol{r}_0', \tag{2.10}$$

if

$$G_{21}(\boldsymbol{r}, \boldsymbol{r}_0; k) = |\boldsymbol{r} - \boldsymbol{r}_0|^{-1} \exp ik\,|\boldsymbol{r} - \boldsymbol{r}_0| - |\boldsymbol{r} - \approx{\boldsymbol{r}}_0|^{-1} \exp ik\,|\boldsymbol{r} - \approx{\boldsymbol{r}}_0|, \tag{2.11}$$

and

$$\overset{\approx}{\boldsymbol{r}}_0 = (x_0, y_0, -z_0 + 2a). \tag{2.12}$$

Proof: Equation (2.1) can be merged into

$$(\nabla_r^2 + k^2)\, K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) = \delta(\boldsymbol{r} - \boldsymbol{r}_0) + k^2(1 - n^2(\boldsymbol{r}))\, K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k). \tag{2.13}$$

The solution of the inhomogeneous Helmholtz eq. (2.13) with inhomogeneity $\delta(\boldsymbol{r} - \boldsymbol{r}_0) + k^2(1 - n^2(\boldsymbol{r}))\, K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k)$ and boundary conditions (2.2) and (2.3) is eq. (2.4) because the function G_{12} is the appropriate Green's function:

$$(\nabla_r^2 + k^2)\, G_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) = \delta(\boldsymbol{r} - \boldsymbol{r}_0) \tag{2.14}$$

$$G_{12}\,(\boldsymbol{r}, \boldsymbol{r}_0; k) = 0, \quad \text{if} \quad \boldsymbol{r}_0 \in z = 0 \quad \forall \boldsymbol{r} \in D, \tag{2.15}$$

$$\lim_{r\to\infty} r \left(\frac{\partial}{\partial r} G_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) - ikG_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) \right) = 0. \tag{2.16}$$

(See also eqs. (1.3) and (1.4)). Eq. (2.10) is derived similarly.

Remark: The Green's function G_{12} and G_{21} are the well-known Rayleigh-Sommerfeld Green's functions connected with the propagation of a scalar wave field in free space from an infinite plane.

Let us assume that

$$n^2(\boldsymbol{r}) = 1 - \varepsilon\delta(\boldsymbol{r} - \boldsymbol{b}), \tag{2.17}$$

if $\boldsymbol{b}$ denotes a point situated between the planes $z = 0$ and $z = a$. The solutions of eqs. (2.4) and (2.10) are for sufficiently small ε excellently approximated by the first term of the Neumann series solution (first iteration) and eqs. (1.7), (1.8), (1.9), (2.4) and (2.10) show that the required surface Green's functions read as:

$$\begin{aligned} &\frac{\partial}{\partial z_0} K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) \\ &= \frac{\partial}{\partial z_0} G_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) + \frac{\partial}{\partial z_0} \varepsilon k^2 G_{12}(\boldsymbol{r}_0, \boldsymbol{b}; k)\, G_{12}(\boldsymbol{b}, \boldsymbol{r}; k) + 0(\varepsilon^2), \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &K_{21}(\boldsymbol{r}, \boldsymbol{r}_0; k) \\ &= \frac{\partial}{\partial z} G_{21}(\boldsymbol{r}, \boldsymbol{r}_0; k)\bigg|_{z=a} + \frac{\partial}{\partial z} \varepsilon k^2 G_{21}(\boldsymbol{r}, b; k)\, G_{21}(\boldsymbol{b}, \boldsymbol{r}_0; k)\bigg|_{z=a} + 0(\varepsilon^2). \end{aligned} \tag{2.19}$$

Inserting eqs. (2.5) and (2.11) into (2.18) and (2.19) leads to:

$$\frac{\partial}{\partial z_0} K_{12}(\boldsymbol{r}, \boldsymbol{r}_0; k) = \frac{\partial}{\partial z_0} \mid \boldsymbol{r} - \boldsymbol{r}_0 \mid^{-1} \exp ik \mid \boldsymbol{r} - \boldsymbol{r}_0 \mid$$

$$+ \varepsilon k^2 \frac{\partial}{\partial z_0} (\mid \boldsymbol{r} - \boldsymbol{b} \mid^{-1} \exp ik \mid \boldsymbol{r} - \boldsymbol{b} \mid - \mid \boldsymbol{r} - \tilde{\boldsymbol{b}} \mid^{-1} \exp ik \mid \boldsymbol{r} - \tilde{\boldsymbol{b}} \mid)$$

$$(\mid \boldsymbol{b} - \boldsymbol{r}_0 \mid^{-1} \exp ik \mid \boldsymbol{b} - \boldsymbol{r}_0 \mid - \mid \boldsymbol{b} - \tilde{\boldsymbol{r}}_0 \mid^{-1} \exp ik \mid \boldsymbol{b} - \tilde{\boldsymbol{r}}_0 \mid \Big|_{z_0=0} + 0(\varepsilon^2), \tag{2.20}$$

and

$$\frac{\partial}{\partial z} K_{21}(\boldsymbol{r}_1 \boldsymbol{r}_0; k) \Big|_{z=a} = \frac{\partial}{\partial z} \boldsymbol{r} - \boldsymbol{r}_0 \mid^{-1} \exp ik \mid \boldsymbol{r} - \boldsymbol{r}_0 \mid$$

$$+ \varepsilon k^2 \frac{\partial}{\partial z} \mid \boldsymbol{r} - \boldsymbol{b} \mid^{-1} \exp ik \mid \boldsymbol{r} - \boldsymbol{b} \mid - \mid \boldsymbol{r} - \approx{\boldsymbol{b}} \mid^{-1} \exp ik \mid \boldsymbol{r} - \approx{\boldsymbol{b}} \mid$$

$$\mid \boldsymbol{b} - \boldsymbol{r}_0 \mid^{-1} \exp ik \mid \boldsymbol{b} - \boldsymbol{r}_0 \mid - \mid \boldsymbol{b} - \approx{\boldsymbol{r}}_0 \mid \exp ik \mid \boldsymbol{b} - \approx{\boldsymbol{r}}_0 \mid \Big|_{z=a} + 0\,(\varepsilon^2). \tag{2.21}$$

Let us assume for mathematical convenience that $k = a^{-5}$. The first terms on the r.h.s. of eqs. (2.20) and (2.21) are equal. This result is to be expected because these terms describe free space propagation: it has to be impossible, from the symmetry of the system that the propagation of a field from plane to plane in free space from the left to the right, differs from the propagation of the field from the right to the left.

However, a straight forward calculation shows, using eq. (2.6) that the Laurent expansion of the second term at the r.h.s. of eq. (2.20) for large values of $\mid a \mid$ begins with $ba^{-2} F^{(1)} (b, \boldsymbol{r}_0)$ and $\lim_{b\to\infty} F^{(1)} (b, \boldsymbol{r}_0) = 0$. The Laurent expansion of the second term at the r.h.s. of eq. (2.21) begins with $-2a^{-2} F^{(2)} (b, \boldsymbol{r}_0)$ and $F^{(2)} (0, \boldsymbol{r}_0) \neq 0$. This result showes that K_{12} and K_{21} are *not* equal to each other.

3. The construction of a non-reciprocal scalar free space Green's function

Suppose that a surface δ-distribution $\delta(\theta - \theta')\, \delta(\varphi - \varphi')$ is located on a sphere with radius a. The resulting field inside the sphere is therefore a solution to the Helmholtz equation

$$(\nabla^2 - k^2)\, G(\boldsymbol{r}, \boldsymbol{r}') = 0, \quad r < a, \quad r' = a, \tag{3.1}$$

subject to

$$G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\theta - \theta')\, \delta(\varphi - \varphi'), \quad \text{if} \quad r = a. \tag{3.2}$$

The function G denotes the so called surface Green's function. Standard techniques, Sommerfeld [4] § 10 and § 28 lead to:

$$G(\boldsymbol{r}, \boldsymbol{r}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_l^m(\theta, \varphi)\, Y_l^{m*}(\theta', \varphi')\, \frac{j_l(kr)}{j_l(ka)}. \tag{3.3}$$

The disturbance at the point $\boldsymbol{r}' = (a, \theta', \varphi')$ due to a δ-surface distribution at $\boldsymbol{r} = (b, \theta, \varphi)$ reads as:

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{h_l^{(1)}(ka)}{h_l^{(1)}(kb)}\, Y_l^m(\theta, \varphi)\, Y_l^{m*}(\theta', \varphi'). \tag{3.4}$$

The eq. (3.3) and (3.4) are in general *not* equal to each other: choosing the radius of the sphere sufficiently close to a value for which a resonance occurs (e.g. $j_l(ka) \simeq 0$), eq. (3.3) shows that the disturbance at $\boldsymbol{r}' = (b, \theta', \varphi')$ can be made arbitrarily large. However, the expression (3.4) is bounded at $\boldsymbol{r} = (a, \theta, \varphi)$ for all values of b. The latter property reflects the well-known fact that the infinite space has no resonances, Sommerfeld [4], i.e. the dirichlet problem always admits a unique solution for the outer region. Hence G and $G^{(1)}$ are *not* reciprocal. The same type of argument applies to the free space propagation of vector fields. Thus, the conjugo-symmetrical property of the Greens tensor of the electromagnetic field:

$$\mathscr{G}(\boldsymbol{r}, \boldsymbol{r}') = \mathscr{G}^{+}(\boldsymbol{r}', \boldsymbol{r}).$$

if $^+$ denotes the adjoint of a tensor, is *not* valid for surface Green's tensors.

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